inverse. This result can also be seen geometrically and a dihedral rotation group may be defined by the fact that it is composed of a cyclic group of order $n$ and $n$ operations of order two transforming each operation of this cyclic subgroup into its inverse. The regular polygon group of order $2n$ is therefore non-commutative whenever $n$ exceeds two.

It is now easy to determine all the quadrangle groups. The largest of these is the group of the square. This consists of the cyclic group of order 4 and four additional operations of order 2 corresponding to the rotations around its four lines of symmetry. Since each of these eight rotations, with the exception of the identity, permutes at least two of the vertices, the square group can be represented on four letters and hence it is simply isomorphic with the octic group.* It contains two non-cyclic subgroups of order 4. The transitive one is the group of a rectangle and the intransitive one is the group of a rhombus. The other subgroup of order 4 is cyclic and it is the group of the spherical quadrangle whose vertices determine a square. All the other quadrangles belong either to the group of period 2 or to the identity.

It is evident that a crossed quadrangle whose vertices coincide with those of a square or rectangle, belongs to the group of order four. Since crossed polygons are not generally studied in elementary geometry we shall exclude them from our consideration. There are two classes of plane convex quadrangles whose group is of order 2; viz., (1) Those whose diagonals intersect at right angles, only one of them being bisected, (2) Those whose unequal diagonals bisect each other obliquely, the general parallelograms. The latter are the only plane quadrangles which permit a non-identical movement without having any line of symmetry.

If a quadrangle is concave, its group can differ from the identity only when the concave vertex is at the center of the circle circumscribing the other three vertices and when these vertices determine an isosceles triangle. As the group of such a quadrangle is of order two, there are just three classes of plane quadrangles which have this group. A spherical quadrangle belongs to this group if the common perpendicular of the lines joining two pairs of vertices bisects each of these lines, and the vertices of the quadrangle do not determine a square. All the other quadrangles admit no movement besides the identity.

There are only two substitution groups on three letters. These are the groups of movements of the plane and the spherical equilateral triangles. There are seven groups on four letters. Of the latter, the group of order 2 is the group of movements of various quadrangles, the three groups of order 4 are the groups of movements of the rectangle, rhombus, and spherical quadrangle whose vertices determine a square, respectively; and the octic group is the group of movements of the square. As the other two possible substitution groups of degree 4 are neither cyclic nor of the dihedral rotation type they cannot be the groups of movements of any polygons. It is, however, easy to verify that they are the groups of movements of the regular tetrahedron and cube respectively. Hence

*Pierpont, Annals of Mathematics, Vol. 1, 1900, page 140.

each one of the seven possible substitution groups on four letters is a group of movements.

Comparatively few of the groups whose degrees exceed 4 are groups of movements. The most interesting of these is the icosaedron group, which is simply isomorphic with the alternating group on five letters, and is the group of movements of the regular icosaedron and also of the regular dodecaedron. The regular octaedron has the same group of movements as the cube; viz., the symmetric group on four letters, while the regular tetraedron admits only the rotations corresponding to the alternating group on four letters. The orders of these groups can be easily found by observing the number of possible rotations of the solid which transform a face into itself and multiplying this number by the number of faces.

If the base of a regular pyramid has $n$ sides ($n \neq 3$) its group is the cyclic group of order $n$. This is also the case when $n=3$ and the pyramid is not a regular tetraedron. If the base of a regular prism has $n$ sides ($n \neq 4$) its group is the dihedral rotation group of order $2n$. When $n=4$ and a lateral edge is not equal to a base edge the group of the regular prism is also the dihedral rotation group of order $2n$. The rectangular parallelopiped with three unequal edges belongs to the rectangle group and it is easy to construct pyramids which belong to the group of order two. Hence *all the possible groups of movements of polygons are also groups of movements of either the pyramids or the prisms.*

It has been observed above that solids may have three additional groups of movements. That is, there are three groups of rotation which are neither cyclic nor of the dihedral rotation type. The proof that there are no more than three such groups is not very difficult. It is only necessary to observe that each complete set of conjugate vertices must have the same group and must lie on a certain sphere, whenever the number of these vertices exceeds two. Hence we may confine our attention to any one complete set of conjugate vertices provided it includes more than two.*

If a complete set of conjugate vertices (which may consist of just one vertex) lies on a plane which does not pass through the center of the minimum circumscribed sphere, the group of the solid is cyclic. When this condition is not satisfied and a complete set of conjugate vertices lies on two parallel planes the group of the solid must be metacyclic.

In all other cases the points where the conjugate axes of rotations meet the surface of the sphere on which the complete set of conjugate vertices lies must form the vertices of a regular polyhedron.† As the number of the former points must exceed two, their group is identical with the group of the vertices. These results may be stated as follows: *If the group of movements of a system of points is neither cyclic nor of the dihedral rotation type, it must be the tetraedron, octahedron, or the icosaedron group.*

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*If each conjugate set is composed of only two vertices it is easy to see that the group must be either of order two or of order four.

†For a more complete proof see Jordan, *Annali di Matematica*, Vol. 2, 1868; also Klein, *Ueber das Ikosaeder*, 1884, page 21.

# THREE ALGEBRAIC NOTES.

By DR. L. E. DICKSON.

## I. Transformation of a Solvable Quintic into Itself.

As well known, the following quintic equation

$$y^5+py^3+\tfrac{1}{5^2}p^2y+r=0 \tag{1}$$

may be solved algebraically by making the substitution

$$y=z-\tfrac{1}{5}p/z,$$

quite analogous to Cartan's solution of a reduced cubic equation.

We here consider another property of (1), which may be written

$$y\{(y^2+\tfrac{1}{2}p)^2-\tfrac{1}{2^0}p^2\}+r=0. \tag{1'}$$

This form suggests the following quadratic transformation

$$y^2+\tfrac{1}{2}p=w,\ y=\sqrt{}(w-\tfrac{1}{2}p).$$

After rationalization, (1') then becomes

$$(w-\tfrac{1}{2}p)(w^2-\tfrac{1}{2^0}p^2)^2=r^2,$$

a solvable quintic containing al powers $\leqq 5$ of $w$. To remove the term $w^4$, set $w=x+\tfrac{1}{10}p$. We thus obtain

$$(x-\tfrac{3}{5}p)(x^3+\tfrac{1}{5}px-\tfrac{1}{2^1\,5}p^2)^2=r^2,$$

$$x^5-\tfrac{1}{5}p^2x^3+\tfrac{1}{1\,2\,5}p^4x-\frac{2}{5^5}p^5-r^2=0, \tag{2}$$

a quintic lacking the term $x^2$ as well as $x^4$. Setting

$$Y=ax,\ P=-\tfrac{1}{5}p^2a^2,\ R=-\frac{2}{5^5}p^5a^5-r^2a^5,$$

equation (2) takes the following form, analogous to (1),

$$Y^5+PY^3+\tfrac{1}{5}P^2Y+R=0. \tag{3}$$

Now equations (1) and (3) are identical in exactly two cases:

$$a=\sqrt{\frac{-5}{p}},\ r=\sqrt{\left(\frac{-p}{5}\right)^5};\ a=-\sqrt{\frac{-5}{p}},\ r=-2\sqrt{\left(\frac{-p}{5}\right)^5}.$$

Theorem. *According as* $r=[\sqrt{(-p/5)}]^5$ *or* $-2[\sqrt{(-p/5)}]^5$, *the substitution* $y^2=Y\sqrt{(-p/5)}-\frac{2}{5}p$, *or* $y^2=-Y\sqrt{(-p/5)}-\frac{2}{5}p$ *transforms the quintic equation* (1) *into itself.*

In the second case, $\frac{r^2}{4}+\left(\frac{p}{5}\right)^5=0$, so that the quintic has two pairs of equal roots. In fact (Dickson's *College Algebra*, page 189), the roots are

$$2\sqrt{(-p/5)},\ \text{two each}\ \alpha\sqrt{(-p/5)},\ \text{two each}\ \beta\sqrt{-p/5)},$$

where $\alpha=\epsilon+\epsilon^4$, $\beta=\epsilon^2+\epsilon^3$ are the roots of $w^2+w-1=0$; also $\epsilon^5=1$.

## II. Reducibility of a Reciprocal Sextic Equation.

The general reciprocal equation of the sixth degree may be written

$$x^6-px^5+qx^4-rx^3+qx^2-px+1=0. \tag{4}$$

As known, its solution may be made to depend upon a cubic and a quadratic equation; thus, set

$$x+\frac{1}{x}=z,\ \text{whence}\ x^2+\frac{1}{x^2}=z^2-2,\ x^3+\frac{1}{x^3}=z^3-3z. \tag{5}$$

Dividing (4) by $x^3$, we then obtain

$$z^3-pz^2+(q-3)z+2p-r=0. \tag{6}$$

The question here proposed is the determination of the conditions under which (4) has a factor of specified form and with coefficients in an initially given domain of rationality. Some examples will make clear what is meant by a *domain*. It is a familiar theorem that, when $p$ is a prime number, $x^{p-1}+\ldots\ldots+x^2+x+1$ is algebraically irreducible, not being expressible as a product of two factors each of the form $f=ax^n+bx^{n-1}+\ldots\ldots+t$, where $1\leqq n<p-1$, and $a$, $b$, ....., $t$ are *integers*. It readily* follows that there is no factor $f$ with *rational* coefficients. Hence $x^{p-1}+\ldots\ldots+x^2+x+1$ is irreducible in the *domain of all rational numbers*. However, it may become reducible in a larger domain. Thus, for $p=5$,

$$x^4+x^3+x^2+x+1=[x^2+\tfrac{1}{2}x(1+\sqrt{5})+1][x^2+\tfrac{1}{2}x(1-\sqrt{5})+1],$$

and hence is reducible in the domain of all rational functions of $\sqrt{5}$ with integral coefficients. It becomes completely reducible into linear factors in a domain containing an imaginary fifth root of unity.

We return to the question of the reducibility of (4) in a given domain or field $F$ containing the coefficients of $p$, $q$, $r$. The condition that (4) shall have a

*For one proof, with references to the various proofs, of these two statements, see the writer's *Introduction to the Theory of Algebraic Equations* (Wiley & Sons, 1903), page 76.

linear factor $x-x_1$ is that the cubic (6) shall have a root $z_1$ in the domain $F$ such that $x^2-z_1x+1=0$ has a root $x_1$ in $F$, the latter requiring that $z_1^2-4$ be the square of an element of $F$. With one linear factor $x-x_1$ is associated a second, viz., $x-1/x_1$. The condition for at least three linear factors is that (6) shall have at least two and hence all its roots $z$ in $F$, such that $z^2-4$ be a square in $F$.

From the above method of solving (4), or by undetermined coefficients, it follows that (4) has a factor of the form $x^2-z_1x+1$ if and only if equation (6) has a root $z_1$ in $F$. Next, if (4) has a factor $x^2-kx+t$, where $t\neq 0$, $t\neq 1$, it has also the factor $t^{-1}(tx^2-kx+1)=x^2-t^{-1}kx+t^{-1}$, and hence a third factor $x^2-zx+1$. The condition for this third factor was that (6) have a root $z$ in $F$. Let this condition be satisfied. Then the quotient of (4) by $x^2-zx+1$ is the reciprocal quartic equation with coefficients in $F$:

$$x^4+(z-p)x^3+(z^2-zp-1+q)x^2+(z-p)x+1=0. \tag{7}$$

We are therefore led to consider the conditions for the factorization

$$x^4-ax^3+bx^2-ax+1=(x^2-kx+t)(x^2-kt^{-1}x+t^{-1}). \tag{8}$$

The conditions are

$$a=k(1+t^{-1}),\ b=t+t^{-1}+k^2t^{-1}. \tag{9}$$

Eliminating* $k$, we obtain the reciprocal condition on $t$

$$t^4+(2-b)t^3+(a^2-2b+2)t^2+(2-b)t+1=0. \tag{10}$$

If (10) has a root $t\neq -1$ in $F$, the first condition (9) determines $k$ in $F$. If every root of (10) equals $-1$, then $a=0$, $b=-2$, $k=0$, so that the factorization (8) is evident. For the general case, we may readily determine the conditions under which (10) has a root in $F$. First, it is necessary that $Z\equiv t+t^{-1}$ belong to $F$. By (10),

$$Z^2-(2-b)Z+a^2-2b=0,\ (Z-1+\tfrac{1}{2}b)^2=(1+\tfrac{1}{2}b)^2-a^2.$$

Hence $(1+\frac{1}{2}b)^2-a^2$ must be the square of an element $l$ of $F$. Next, from $t^2-tZ+1=0$, $Z^2-4\equiv\frac{1}{2}b^2-2-a^2+l(2-b)$ must be a square in $F$.

To proceed otherwise, let $z_1$ and $z_2$ be the roots of $z^2-az+b-2=0$. Then

$$x^4-ax^3+bx^2-ax+1=(x^2-z_1x+1)(x^2-z_2x+1).$$

Setting $Z_1=\sqrt{(z_1^2-4)}$, $Z_2=\sqrt{(z_2^2-4)}$, the four roots are

$$\tfrac{1}{2}z_1+\tfrac{1}{2}Z_1,\ \tfrac{1}{2}z_1-\tfrac{1}{2}Z_1,\ \tfrac{1}{2}z_2+\tfrac{1}{2}Z_2,\ \tfrac{1}{2}z_2-\tfrac{1}{2}Z_2.$$

*Upon eliminating $t$, there results a non-reciprocal quartic for $k$.

The two roots of either of the quadratic factors (8) are not reciprocal. By proper choice of the signs of $Z_1$ and $Z_2$, we may assume that $\frac{1}{2}z_1+\frac{1}{2}Z_1$ and $\frac{1}{2}z_2+\frac{1}{2}Z_2$ are the roots of one of the quadratic factors (8) and the remaining two the roots of the other. Now the coefficients of a quadratic (8) belong to $F$ if and only if the sum and product of its two roots do. But $z_1+z_2=a$ and $z_1z_2=b-2$ belong to $F$. The conditions are therefore that

$$Z_1+Z_2,\ z_1Z_2+z_2Z_1+Z_1Z_2,\ -z_1Z_2-z_2Z_1+Z_1Z_2$$

shall belong to $F$. It follows first that $Z_1Z_2\equiv 2l/[(1+\frac{1}{2}b)^2-a^2]$ shall belong to $F$, say $=-2l$. Next, $z_1Z_2+z_2Z_1$ must belong to $F$. Its square equals

$$2z_1^2z_2^2-4z_1^2-4z_2^2+2z_1z_2Z_1Z_2=\tfrac{1}{4}[\tfrac{1}{2}b^2-2-a^2+l(2-b)].$$

Hence $\frac{1}{2}b^2-2-a^2+l(2-b)$ must be a square in $F$. We have now obtained the two conditions given earlier. I remains to verify that $Z_1+Z_2$ belong to $F$. But

$$(z_1+z_2)(Z_1+Z_2)=(z_1Z_1+z_2Z_2)+(z_1Z_2+z_2Z_1),$$

$$(z_1Z_1+z_2Z_2)(z_1Z_2+z_2Z_1)=z_1z_2(Z_1^2+Z_2^2)+Z_1Z_2(z_1^2+z_2^2),\ \text{in } F.$$

Consider finally the most interesting case of all, that in which the sextic (4) has an irreducible cubic factor. Call its roots $\kappa$, $\lambda$, $\mu$. No two of the three are reciprocal, since the product of all three would then equal the third root, which would thus belong to $F$, making the cubic reducible. Hence $\kappa^{-1}$, $\lambda^{-1}$, $\mu^{-1}$ are the roots of a second cubic factor of (4). Hence (4) is the product of the two cubic factors

$$(11)\qquad x^3+ax^2+bx+c,\ x^3+\frac{b}{c}x^2+\frac{a}{c}x+\frac{1}{c}.$$

Equating the coefficients of the product to those of (4), we get

$$(12)\qquad a+\frac{b}{c}=-p,\ b+\frac{ab}{c}+\frac{a}{c}=q,\ c+\frac{1}{c}+\frac{b^2}{c}+\frac{a^2}{c}=-r.$$

Substituting the value of $a$ from the first in the last two, we get

$$(13)\qquad b^2+b(1+pc-c^2)+pc+qc^2=0,$$

$$(14)\qquad b^2(1+c^2)+2bpc+p^2c^2+c^2+c^4+rc^3=0.$$

Eliminating $b^2$, we get $bU+W=0$, where

$$(15)\qquad U=c^4-pc^3+pc-1,\ W=p^2c^2+c^3+c^4+rc^3-pc-pc^3-qc^2-qc^4.$$

Substituting $b=-W/U$ in (13) and cancelling the factor $c^2$, we get

(16) $c^8+1+(c^7+c)(r-2p)+(c^6+c^2)(q^2-2pr+2p^2+1-2q)$

$$+(c^5+c^3)(r-2p)(p^2-2q+1)+c^4(2q^2+r^2+p^4+4p^2-4q-4p^2q)=0,$$

which is evidently a reciprocal equation. Making the substitution, similar to (5),

(17) $$c+\frac{1}{c}=y,\ c^2+\frac{1}{c^2}=y^2-2,\ c^3+\frac{1}{c^3}=y^3-3y,\ c^4+\frac{1}{c^4}=y^4-4y^2+2,$$

equation (16) becomes

(18) $$y^4+y^3(r-2p)+y^2(q^2-2pr+2p^2-3-2q)$$

$$+y(r-2p)(p^2-2q-2)+p^4-4p^2q+4pr+r^2=0.$$

Suppose that $y$ and then $c$ can be determined in $F$. Now

(19) $$U=(c^2-1)(c^2-pc+1).$$

If $U\neq 0$, then $b$ and $a$ are uniquely determined by $bU+W=0$ and the first relation (12). Next, $U=0$ if and only if $c=\neq 1$ or $c^2+1=cp$, whence $y=2,\ -2$, or $p$. If $c=1$, $W=0$ gives $r-2q+p^2-2p+2=0$, and (18) reduces to the square of the preceding, while (13) and (14) reduce to $b^2+bp+p+q=0$. If $c=-1$, $W=0$ gives $r+2q-2p-p^2-2=0$, and (18) reduces to the square of the preceding, while (13) and (14) reduce to $b^2-bp-p+q=0$. If $c^2=pc-1$, whence $y=p$, then $W=0$ gives $r+p-pq=0$, and (18) reduces to $(r+p-pq)^2=0$, while (13) and (14) reduce $b^2+2b+qc^2+pc=0$.

*In general, the values of a and b follow rationally when c is found from the quartic and quadratic. In the three cases mentioned, c is given, but b requires the solution of a quadratic.*

## III. The General Term of a Recurring Series.

The usual examples under this topic lead to a generating fraction which can be resolved into partial fractions with binomial denominators, so that the general term of the expansion into series is readily found. It is otherwise with the following example (No. 4, page 148, Dickson's *College Algebra*):

(20) $$1+6x-4x^2-40x^3-112x^4-32x^5+704x^6+\ldots.$$

The generating relation is $u_n-2xu_{n-1}+4x^2u_{n-2}+8x^3u_{n-3}=0$; the generating fraction is

(21) $$(1+4x-12x^2)/D,\ D\equiv 1-2x+4x^2+8x^3.$$

*First method.* Set $D=(1-2\alpha x)(1-2\beta x)(1-2\gamma x)$. Then

(22) $$\alpha+\beta+\gamma=1,\ \alpha\beta+\alpha\gamma+\beta\gamma=1,\ \alpha\beta\gamma=-1.$$

The discriminant $\Delta \equiv (\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\alpha)^2$ is found to be $-44$. Now

$$\frac{1}{D} = \frac{a}{1-2\alpha x} + \frac{b}{1-2\beta x} + \frac{c}{1-2\gamma x},$$

$$1 = a\left(1-\frac{\beta}{\alpha}\right)\left(1-\frac{\gamma}{\alpha}\right) = b\left(1-\frac{\alpha}{\beta}\right)\left(1-\frac{\gamma}{\beta}\right) = c\left(1-\frac{\alpha}{\gamma}\right)\left(1-\frac{\beta}{\gamma}\right).$$

$$\therefore \frac{-\Delta}{D} = \frac{a^2(\alpha-\beta)(\beta-\gamma)^2(\gamma-\alpha)}{1-2\alpha x} + \frac{\beta^2(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)^2}{1-2\beta x}$$

$$+ \frac{\gamma^2(\alpha-\beta)^2(\beta-\gamma)(\gamma-\alpha)}{1-2\gamma x}.$$

Expanding $1/(1-2\alpha x)$, ....., into series, we obtain as the coefficient of $2^r x^r$:

$$C_r = \alpha^{r+2}(\alpha-\beta)(\beta-\gamma)^2(\gamma-\alpha) + \beta^{r+2}(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)^2$$

$$+\gamma^{r+2}(\alpha-\beta)^2(\beta-\gamma)(\gamma-\alpha).$$

In this the coefficient of $\alpha^{r+2}$ equals

$$3\beta^2\gamma^2 - (\alpha^2\beta^2+\alpha^2\gamma^2+\beta^2\gamma^2) + 3\alpha^2\beta\gamma - \alpha\beta\gamma(\alpha+\beta+\gamma) + \alpha\beta^3 + \alpha\gamma^3 - \beta\gamma^3 - \beta^3\gamma$$

$$= 3\beta^2\gamma^2 + \beta\gamma - \alpha^4 - 9\alpha - 2,$$

in view of (22) and the consequent relations

$$\alpha^2\beta^2+\alpha^2\gamma^2+\beta^2\gamma^2 = (\alpha\beta+\alpha\gamma+\beta\gamma)^2 - 2\alpha\beta\gamma(\alpha+\beta+\gamma) = 3,$$

$$\alpha^3+\beta^3+\gamma^3 = (\alpha+\beta+\gamma)^3 - 3(\alpha+\beta+\gamma)(\alpha\beta+\alpha\gamma+\beta\gamma) + 3\alpha\beta\gamma = -5,$$

$$\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\alpha\gamma+\beta\gamma) = -1.$$

Hence, if we set $S_n = \alpha^n+\beta^n+\gamma^n$, we obtain

$$C_r - 8S_r - S_{r+1} - 2S_{r+2} - 9S_{r+3} - S_{r+6}. \tag{23}$$

Since $\alpha$, $\beta$, $\gamma$ are the roots of $y^3-y^2+y+1$, we have Newton's identity

$$S_k - S_{k+1} + S_{k-2} + S_{k-3} = 0 \quad (k \geqq 3). \tag{24}$$

Hence

$$S_k + 2S_{k-3} + S_{k-4} = 0. \tag{25}$$

Hence (23) becomes

$$C_r = 10S_r + 6S_{r+1} - 8S_{r+2}. \tag{26}$$

Now $C_r$ denoted the coefficient of $2^r x^r$ in the expansion of $44/D$. Hence the coefficient of $2^{r+2}x^{r+2}$ in the expansion of (21) is

$$\tfrac{1}{44}(C_{r+2}+2C_{r+1}-3C_r)=\tfrac{1}{44}(36S_{r+2}+28S_{r+1}-12S_r),$$

the last reduction following from (24) and (26). *Hence the coefficient of $x^{r+2}$ in the series* (20) *is* $\frac{1}{11}(9S_{r+2}+7S_{r+1}-3S_r)2^{r+2}$.

By way of check, we note that $S_1=1$, $S_2=-1$, $S_3=-5$, as shown above. Hence $S_4=-5$, $S_5=1$, by (24). The coefficients of $x^3$ and $x^4$ are therefore

$$\tfrac{1}{11}(-45-7-3)2^3=-40,\ \tfrac{1}{11}(-45-35+3)2^4=-112.$$

Applying Waring's formula (Serret, *Algèbre*, 1, page 449) to the cubic $y^3-y^2+y+1=0$, we obtain the explicit result

$$(27)\qquad S_n=\sum\frac{(-1)^{\lambda_2+\lambda_3}n(\lambda_1+\lambda_2+\lambda_3)!}{(\lambda_1+1)!\,(\lambda_2+1)!\,(\lambda_3+1)!},\qquad \begin{pmatrix}\text{summed for all integers}\geqq 0\\ \text{such that } \lambda_1+2\lambda_2+3\lambda_3=n\end{pmatrix}.$$

*Second method.* Equate expression (21) to $a_0+2a_1x+2^2a_1x^2+\ldots\ldots+2^ra_rx^r+\ldots\ldots$, clear of fractions, and compare coefficients. Then

$$1=a_0,\ 2=-a_0+a_1,\ -3=a_0-a_1+a_2,\ 0=a_0+a_1-a_2+a_3,\ \ldots\ldots,$$

$$0=a_r+a_{r+1}-a_{r+2}+a_{r+3},\quad (r=0,\ 1,\ 2,\ \ldots\ldots).$$

Noting that the determinant of the coefficients of the first $n$ equations is 1, we get

$$(28)\qquad a_n=\begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots\ldots & 0 & 1\\ -1 & 1 & 0 & 0 & 0 & 0 & \ldots\ldots & 0 & 2\\ 1 & -1 & 1 & 0 & 0 & 0 & \ldots\ldots & 0 & -3\\ 1 & 1 & -1 & 1 & 0 & 0 & \ldots\ldots & 0 & 0\\ 0 & 1 & 1 & -1 & 1 & 0 & \ldots\ldots & 0 & 0\\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots\\ 0 & \ldots\ldots & 0 & 1 & 1 & -1 & 1 & 0 & 0 & 0\\ 0 & \ldots\ldots & 0 & 0 & 1 & 1 & -1 & 1 & 0 & 0\\ 0 & \ldots\ldots & 0 & 0 & 0 & 1 & 1 & -1 & 1 & 0\\ 0 & \ldots\ldots & 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 \end{vmatrix}.$$

Expanding according to the elements of the last column, we get

$$(29)\qquad a_n=(-1)^n(D_n-2D_{n-1}-3D_{n-2}),$$

where $D_n$ denotes the minor obtained upon deleting the first row and first column of (28). Adding the first row to the second in $D_n$, we get

$$D_n=\begin{vmatrix} -1 & 1 & 0 & 0 & 0 & 0 \cdots 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \cdots 0 \\ 1 & 1 & -1 & 1 & 0 & 0 \cdots 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \cdots 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \cdots 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} = -\begin{vmatrix} -1 & 1 & 0 & 0 & 0 & 0 \cdots \\ 1 & 1 & 1 & 0 & 0 & 0 \cdots \\ 0 & 1 & -1 & 1 & 0 & 0 \cdots \\ 0 & 0 & 1 & -1 & 1 & 0 \cdots \\ 0 & 0 & 1 & 1 & -1 & 1 \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$

$$= +\begin{vmatrix} 2 & 1 & 0 & 0 & 0 & 0 \cdots \\ 1 & -1 & 1 & 0 & 0 & 0 \cdots \\ 0 & 1 & -1 & 1 & 0 & 0 \cdots \\ 0 & 1 & 1 & -1 & 1 & 0 \cdots \\ 0 & 0 & 1 & 1 & -1 & 1 \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix},$$

where the second determinant was reduced by adding its first row to its second. Expanding the third determinant according to its first column,

$$D_n=2D_{n-3}-D_{n-4}. \tag{30}$$

For $d_r=(-1)^r D_r$, relations (29) and (30) become

$$a_n=d_n+2d_{n-1}-3d_{n-2},\ d_n=-2d_{n-3}-d_{n-4}. \tag{31}$$

The second recursion relation is precisely recursion relation (25).

*Third method.* The existence of the three-term recursion relation, just mentioned, suggested the following method, which aside from its artificiality is perhaps the most satisfactory one of the three. We employ the identity

$$(1-2x+4x^2+8x^3)(1+2x)=1+16x^3+16x^4.$$

Let ${}_rC_k$ denote the number of combinations of $r$ things $k$ at a time. Then

$$\frac{1}{1+16x^3+16x^4}=\sum_{r=0}^{\infty}(-1)^r\sum_{k=0}^{r}{}_rC_k\,(16x^3)^k(16x^4)^{r-k}=\sum_{r,k}(-1)^r\,{}_rC_k\,16^r x^{4r-k}.$$

The coefficient of $x^n$ is therefore

$$K_n=\sum_r(-1)^r\,{}_rC_{4r-n}16^r=\sum_{r=0}^{n'}(-1)^r 16^r\,{}_rC_{n-3r}, \tag{32}$$

where $n'$ is the greatest integer in $n/3$. Now expression (21) equals

$$\frac{(1+4x-12x^2)\,(1+2x)}{(1-2x+4x^2+8x^3)\,(1+2x)}=\frac{1+6x-4x^2-24x^3}{1+16x^3+16x^4}.$$

*Hence the coefficients of $x^n$ in series* (20) *is* $K_n+6K_{n-1}-4K_{n-2}-24K_{n-3}$.

# A SIMPLE EXISTENCE-PROOF FOR LOGARITHMS.

By JOHN WESLEY YOUNG, Instructor in Mathematics, Northwestern University.

Professor Osgood in an introduction to a recent paper by Mr. Bradshaw* says: "The student of mathematics and physics meets logarithms for the first time at an early stage. He is told that 'the logarithm of a number is the exponent of the power to which a certain number, taken as the base, must be raised in order to equal the given number.' The definition is purely formal. Probably the beginner has never seen a proof for the existence even of the fifth root of 2, ....., and if he has, it is not likely that it has meant anything to him. He tacitly assumes that every number has a positive $q$th root, $q$ being any positive integer. It is then an easy step to any rational power, and the irrational powers are thought of as limiting cases, the principle being that, whenever one wants a limit in mathematics, the limit exists.

"Now all of these assumptions have been justified by rigorous $\varepsilon$-proofs in well known treatises on modern analysis. But the general student of mathematics and physics does not read these proofs, for they are uninteresting to him; and thus the great majority of students of the Calculus never see a proof that there is such a thing as a logarithm."

Mr. Bradshaw, in the paper cited, gives a proof by means of elementary calculus. In the following we give a proof which, while rigorous, makes use of only elementary algebra and which is much shorter than the proofs usually given.†

But first let us state precisely what is required, from an arithmetic point of view, to prove, the existence of such numbers as rational and irrational powers, logarithms, etc. Cantor's definition of an irrational number is as follows.‡ Suppose we have a series of rational numbers

$$\phi_0, \phi_1, \phi_2, \ldots, \phi_n, \ldots \tag{1}$$

with the property that for every number $\varepsilon > 0$ we may find a number $\mu$ such that the relation

$$| \phi_{n+r} - \phi_n | < \epsilon$$

is satisfied, provided only that $n$ be greater than $\mu$, $r$ being any positive integer; then if $\lim_{n \doteq \infty} [\phi_n]$ is not a rational number, the series (1) *defines* an irrational number equal to $\lim_{n \doteq \infty} [\phi_n]$. In order to prove that a certain irrational number $a$ *exists* it is necessary to establish a series (1) such that $| a - \phi_n | < \epsilon$, provided only that $n > \mu$.

*J. W. Bradshaw, The Logarithm as a Direct Function; *Annals of Mathematics* (January, 1903), 2nd series, v. 4, p. 51.

†Stolz, in his *Allgemeine Arithmetik*, devotes 24 pages to the discussion of powers, roots and logarithms; cf. Vol. I, Chap. 8, pp. 125-148.

‡Stolz, *Allgemeine Arithmetik*, I, p. 105.

We now turn to the problem of proving the existence of a logarithm of any positive number. Of irrational operations we shall assume only the possibility of extracting the square root of any positive number; that such a root exists may be seen geometrically or (better) the ordinary $\epsilon$-proof may be supplied, which in this case is very simple.* Consider two sequences of numbers $G_0$ and $A_0$, the former in geometric, the latter in arithmetic progression:

$$(G_0) \quad \ldots, a^{-m}, \ldots, a^{-2}, a^{-1}, 1, a, a^2, \ldots, a^n, \ldots$$

$$(A_1) \quad \ldots, -mb, \ldots, -2b, -b, 0, b, 2b, \ldots, nb, \ldots$$

where $a$, $b$ are any positive numbers ($a \neq 1$). Establish a one-to-one correspondence between the terms of the sequences, such that 1 of $G_0$ corresponds to 0 of $A_0$, $a$ of $G_0$ to $b$ of $A_0$, and in general $a^k$ of $G_0$ to $kb$ of $A_0$, for all integral, positive or negative, values of $k$. Then if $p$, $q$ be any two numbers of $G_0$, and if $\phi(p)$, $\phi(q)$ denote the corresponding numbers of $A_0$, we have

$$\phi(pq) = \phi(p) + \phi(q). \tag{2}$$

It is our object to prove the existence of numbers $\phi(x)$, which have the property (2), for all real positive values of $x$. To do this insert a geometric mean between some pair of consecutive numbers of $G_0$, and corresponding thereto insert an arithmetic mean between the corresponding pair of $A_0$, and suppose this done for every pair of consecutive numbers of the two sequences. We thus obtain two new corresponding sequences $G_1$, $A_1$, such that if now $p$, $q$ be any two numbers of $G_1$ and $\phi(p)$, $\phi(q)$ the corresponding numbers of $A_1$, relation (2) still holds. By treating $G_1$, $A_1$ as we did $G_0$, $A_0$, we obtain two new sequences $G_2$, $A_2$; and this process may be continued *ad libitum*. After $n$ steps we have two sequences $G_n$, $A_n$, of which the first has a common ratio equal to $\sqrt[2^n]{a}$ and the second a common difference $b/n$. The latter approaches 0 as a limit when $n$ increases indefinitely; whereas the former with increasing $n$ must approach the limit 1. We have then

$$\sqrt[2^n]{a} = 1 + \varepsilon_n,$$

where $\varepsilon_n$ may be made smaller than any assigned positive number by taking $n$ sufficiently large.

Now let $x$ be any (finite) positive real number whatever. Either $x$ is contained in some $G_n$, and then $\phi(x)$ is already defined as the corresponding number of $A_n$; or $x$ is not contained in any $G_n$ no matter how large we take $n$. In the latter case suppose $x$ lies between the two consecutive numbers $x_n$ and $x_n(1+\varepsilon_n)$ of $G_n$; then we have

$$x - x_n < x_n \varepsilon_n.$$

*Cf. e. g. Fischer and Schwatt, *Higher Algebra*, p. 267, or Stolz, *loc. cit.* p. 129.

Since $\epsilon_n$ can be made as small as desired by taking $n$ sufficiently large, it follows that as $n$ increases indefinitely the sequence

$$x_0, x_1, x_2, \ldots, x_n$$

approaches $x$ as a limit. From a fundamental theorem on limits it follows that the corresponding sequence

$$\phi(x_0), \phi(x_1), \phi(x_2), \ldots, \phi(x_n)$$

approaches a limit, which we *define* as $\phi(x)$.

The function $\phi(x)$, $=y$ say, is now defined for all values of $x$ such that $0<x<+\infty$. Moreover $y$, $=\phi(x)$, *is clearly a one-valued and continuous function in this interval, satisfying the functional relation* (2).

Similar reasoning in the opposite direction (*i. e.* from the arithmetical progression to the geometric) shows that *the inverse function* $x=\phi^{-1}(y)=\psi(y)$, *say, is one-valued and continuous in the interval* $-\infty<y<+\infty$ *and satisfies the functional relation* $\psi(\mu)\psi(\nu)=\psi(\mu+\nu)$.

We must now prove that $\phi(x)$ is identical with $\log_k(x)$, or, what is the same, that $\psi(y)$ is identical with $k^y$, where the base $k$ is $\phi(1)$.*

From (2) we obtain readily

$$\phi(x^m)=m\phi(x) \tag{3}$$

when $m$ is any positive integer. From this we may show that *every positive number a has just one mth root.* For, if $\phi(a)=b$, the number $x=\psi(b/m)$ is easily shown to satisfy the relation $x^m=a$, and conversely the only positive number $x$ satisfying this relation is $\psi(b/m)$.†

Further, if the *meaning* of $a^a$, where $a$ is any rational number (positive, negative, or zero), be defined as usual, this proves the *existence* of all rational powers of $a$.

Now, (3) *holds for all rational values of* $m$. For, replacing $x$ by $x^{1/t}$ and $m$ by $t$ in (3) we obtain

$$\phi(x^{1/t})=(1/t)\phi(t)$$

and thence easily

$$\phi(x^{s/t})=(s/t)\phi(x),$$

$s$, $t$ being integers, and $s$ positive, which is no restriction. By replacing $x$ and $m$ by $k=\psi$ (1) and $y$ respectively in (3), we obtain

$$\phi(k^y)=y\phi(k)=y, \text{ whence } k^y=\psi(y)$$

for all rational values of $y$. If the sequence of rational numbers

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*Cf. in this connection, Tannery, *Theorie des functions d'une variable*, p. 120, §82.

†Bradshaw, *loc. cit.*, page 58.

$$y_0, y_1, y_2, \ldots, y_n$$

approaches a limit $y$, when $n$ increases indefinitely, the sequence

$$\psi(y_0), \psi(y_1), \psi(y_2), \ldots, \psi(y_n)$$

approaches the limit $\psi(y)$, and hence *the identical sequence*

$$k^{y_0}, k^{y_1}, k^{y_2}, \ldots k^{y_n}$$

*approaches a limit.** This limit is *defined* as $k^y$.† Hence for all real values of $y$

$$\psi(y)=k^y \text{ and hence } \phi(x)=\log_k(x).$$

EVANSTON, ILLINOIS, *September 21, 1903.*

## CONVERSE AND OPPOSITE PROPOSITIONS.

By C. M. HIMEL, Baker-Himel School, Knoxville, Tenn.

Wentworth, in the revised edition of his Plane Geometry, page 5, makes the following confusing statements:

"If a direct proposition and its opposite are true, the converse proposition is true; and if a direct proposition and its converse are true, the opposite proposition is true."

"Thus, if it were true that

1. If an animal is a horse, the animal is a quadruped;

2. If an animal is not a horse, the animal is not a quadruped;

it would follow that

3. If an animal is a quadruped, the animal is a horse.

Moreover, if 1 and 3 are true, then 2 would be true."

The statements should read: *Whether a direct proposition is true or not, if the converse is true the opposite is true; if the opposite is true the converse is true.*

Thus, in the above example, if 1 be true, then 3 would follow; if 3 be true, then 2 would be true. Consider the general case:

1. Proposition: If $a$ is $b$, then $c$ is $d$.

2. Converse: If $c$ is $d$, then $a$ is $b$.

3. Opposite: If $a$ is not $b$, then $c$ is not $d$.

We may prove that if either of the last two, irrespective of the first, is true, the other is also true.

First. Suppose you know that the converse is true, and you wish to prove

*This required proof.

†Cf. Stolz, *loc. cit.*, p. 138; Tannery, *Theorie des fonctions d'une variable*, p. 114.

the opposite. You are given that $a$ is not $b$, and you wish to prove that $c$ is not $d$. It is not, because, if it is, then $a$ is $b$; but $a$ is not $b$, therefore $c$ is not $d$.

Second. By a similar proof, if the opposite be true, the converse is true.

For example, from the theorem [compare the proof in Wentworth's Revised Geometry, p. 70, Ex. 55]: If two angles of a triangle are not equal, their bisectors are not equal, one may conclude: If the bisectors of two angles of a triangle are equal, the triangle is isosceles. The proof of the first theorem is, indeed, a proof of the second.

# DEPARTMENTS.

## SOLUTIONS OF PROBLEMS.

### GEOMETRY.

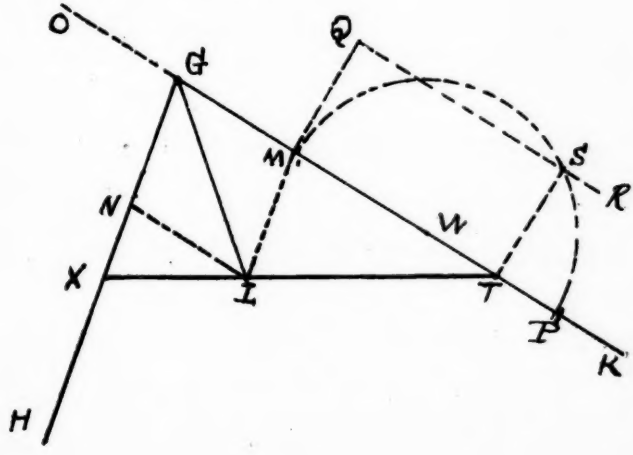
**203. Proposed by W. J. GREENSTREET, A. M., Editor of The Mathematical Gazette, Stroud; England.**

Show that two parabolae can always be drawn through the vertices of a triangle to touch its circumcircle at a vertex, and that the axes of these pairs of curves are orthogonal. Show that any triangle may be circumscribed by a conic so that the tangents at each vertex is parallel to the opposite side.

No solution has been received.

**204. Proposed by ELMER SCHUYLER, B. Sc., Professor of German and Mathematics, Boys' High School, Reading, Pa.**

Construct a triangle, having given an angle, the length of its bisector, and the sum of the including sides. (Phillips and Fisher).



Solution by G. I. HOPKINS, J. SCHEFFER, and G. B. M. ZERR.

Let $AB$ be the sum of the two sides, $CD$ the bisector, and $F$ the given angle. Make $\angle HGK=\angle F$ and bisector $GL=CD$. Draw $LM$ parallel to $HG$ and $NL$ parallel to $GK$. Extend $GK$ making $GO=NG$. Make $OP=AB$, and on $MP$ draw the semicircle $MSP$. Draw the perpendicular $MQ=ML$, also $QR$ parallel to $MP$ and $ST$ perpendicular to $MP$. Through $L$ draw $TX$, then $TGX$ is the required triangle; for $TS^2=MT.PT=QM^2=ML^2$. From similar triangles $XNL$ and $LMT$, $XN:LM::NL:MT$. Since $MLNG$ is a rhombus $LM=NL$. $\therefore XN.MT=ML^2$. $\therefore MT.TP=MT.XN$, whence $NX=TP$. $\therefore GX+GP=OP=AB$.

## CALCULUS.

168. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

The tangent of what Cartesian curve makes an $x$-intercept always $m$ times as long as the corresponding $y$-intercept.

Solution by J. SCHEFFER.

Let the equation of the tangent be $y-y'=\frac{dy'}{dx'}(x-x')$. Consequently the $x$-intercept $=x'-y'\frac{dx'}{dy'}$, and the $y$-intercept$=y'-x'\frac{dy'}{dx'}$; therefore, omitting the accents, by the condition imposed upon the problem

$$-\frac{y}{p}+x=m(y-px);\quad \left(p=\frac{dy}{dx}\right),$$

whence $mxp^2-(my-x)p-y=0$; or, arranged differently, $(px-y)(mp+1)=0$, whence $px-y=0$ and $mp+1=0$. From the former of these two equations we get $y=ax$; and from the second $my+x=b$, where $a$ and $b$ are arbitrary constants. Both equations represent straight lines, the first one of which passes through the origin.

## MECHANICS.

159. Proposed by J. E. SANDERS, Hackney, Ohio.

Required the time for a tree, considered as a material line of uniform density, length $a=100$ feet, to fall; the tree being inclined $\phi=1'$ from perpendicular.

Solution by the PROPOSER.

From Mechanics we find that

$$\frac{\sqrt{(2gh)}t}{l}=\int\frac{d\theta}{\sqrt{[1-(2l/h)\sin^2\frac{1}{2}\theta]}}\ldots\ldots(1),$$

where $g$=gravity=32, $l$=length to center of oscillation=$\frac{2}{3}a$, $h$=height of zero velocity=$2l$ without a sensible error, $t$=time, $\theta$=angle counted from the down point. Reducing, we have,

$$t=\tfrac{1}{2}\sqrt{\left(\frac{l}{g}\right)}\int_{\frac{1}{2}\pi}^{\pi-\phi}\frac{d\theta}{\cos\frac{1}{2}\theta}=\sqrt{\left(\frac{l}{g}\right)}\left[\log\tan(\tfrac{1}{4}\pi+\tfrac{1}{4}\theta)\right]_{\frac{1}{2}\pi}^{\pi-\phi}$$

$$=\sqrt{\left(\frac{l}{g}\right)}\log\frac{\cot\frac{1}{4}\phi}{\tan\frac{3}{8}\pi}=\sqrt{\left(\frac{l}{g}\right)}(8.647491).$$

But $l=\frac{2}{3}$ of 100. $\therefore t=\sqrt{(\frac{100}{48})}(8.647491)=12.481$ seconds.

## AVERAGE AND PROBABILITY.

**139. Proposed by L. C. WALKER, A.M., Professor of Mathematics, Colorado School of Mines, Golden, Col.**

Four points are taken at random on the surface of a given sphere; find the average volume of the tetrahedron formed by the planes passing through the points taken three and three.

No solution has been received.

**140. Proposed by L. C. WALKER, A.M., Professor of Mathematics, Colorado School of Mines, Golden, Col.**

Obtain the average area of a triangle formed by a tangent to the four-cusped hypocycloid and the co-ordinate axes.

**Solution by J. E. SANDERS, Hackney, Ohio.**

The curve's equation is $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$. The length of the perpendicular from the origin on the tangent is

$$\frac{xdx-ydy}{\sqrt{(dx^2+dy^2)}}=\frac{x^{\frac{2}{3}}+y^{\frac{2}{3}}}{\sqrt{(y^{-\frac{2}{3}}+x^{-\frac{2}{3}})}}=\sqrt[3]{(axy)}.$$

The equation to the tangent is $x'/x^{\frac{1}{3}}+y'/y^{\frac{1}{3}}=a^{\frac{2}{3}}$, and its length between the axis=$a^{\frac{2}{3}}\sqrt{(x^{\frac{2}{3}}+y^{\frac{2}{3}})}=a$.

$\therefore$The area of the triangle is $A=\frac{1}{2}a\sqrt[3]{(axy)}=\frac{1}{2}a^2\sin\theta\cos\theta$, if $x=a\cos^3\theta$, and $y=a\sin^3\theta$.

Then, if the tangent is through any point on the curve, the average area is

$$\triangle=\frac{1}{s}\int A\,ds.$$

But $ds=a^{\frac{1}{3}}x^{-\frac{1}{3}}dx=3a\sin\theta\cos\theta\,d\theta$ and $s=\frac{3}{2}a$.

$\therefore\triangle=a^2\int_0^{\frac{1}{2}\pi}\sin^2\theta\cos^2\theta\,d\theta=\frac{1}{16}\pi a^2$, or, if the tangent is to be a random line, the average area is

$$\triangle_1 = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} a^2 \sin\theta \cos\theta \, d\theta = \frac{a^2}{2\pi}.$$

As the problem is stated it seems that either of the above suppositions are allowable.

Also solved by the *PROPOSER*.

142. Proposed by ARTEMAS MARTIN, A. M., Ph. D., LL. D., Washington, D. C.

Two points are taken at random in the arc of a semi-circle, and a third point anywhere in its base. Find the probability that the triangle formed by them is acute.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, West Va.

Let $A$, $B$ be the random points on the arc of the semi-circle, $MN$ its diameter or base, $C$ its center. On $AB$ as diameter describe a circle with center $F$. At $A$, $B$, perpendicular to $AB$, draw $AD$, $BE$ meeting $MN$ in $D$ and $E$. From $A$ draw $AL$ parallel to $MN$ meeting $BE$ in $L$, and from $F$ draw $FG$ perpendicular to $MN$ meeting $MN$ in $G$. If the circle on $AB$ does not intersect $MN$, the third point can be anywhere on $DE$; if the circle on $AB$ intersects $MN$, then the third point can be anywhere on $DE$ without this circle.

$\therefore$ If the circle on $AB$ intersects $MN$ in $H$, then the third point $K$ can be anywhere on $DH$ or $EK$. Let $CA=CB=r$, $\angle MCF=\theta$, $\angle ACF=\varphi$. Then $AB=2r\sin\varphi$, $AL=DE=2r\sin\varphi\operatorname{cosec}\theta=Q$, $FC=r\cos\varphi$, $FG=r\cos\varphi\sin\theta$. When $FG=FB$ we get $r\sin\varphi=r\cos\varphi\sin\theta$.

$\therefore \tan\varphi=\sin\theta$, $HG=\sqrt{(HF^2-FG^2)}=\sqrt{(AF^2-FG^2)}$.

$\therefore HG=r\sqrt{(\sin^2\varphi-\sin^2\theta\cos^2\varphi)}$.

$\therefore DH+EK=DE-2HG=2r[\sin\varphi\operatorname{cosec}\theta-\sqrt{(\sin^2\varphi-\sin^2\theta\cos^2\varphi)}]=Q'$.

Let $\varphi=\tan^{-1}(\sin\theta)=\varphi'$, $p$=required chance.

$$\therefore p = \frac{\left[\frac{1}{2r}\int_0^{\frac{1}{2}\pi}\int_0^{\phi'} Q\,d\theta\,d\varphi + \frac{1}{2r}\int_0^{\frac{1}{2}\pi}\int_{\phi'}^{\theta} Q'\,d\theta\,d\varphi\right]}{\int_0^{\frac{1}{2}\pi}\int_0^{\theta} d\theta\,d\varphi}$$

$$= \frac{8}{\pi^2}\int_0^{\frac{1}{2}\pi}\left[\int_0^{\phi'} \sin\varphi\operatorname{cosec}\theta\,d\varphi + \int_{\phi'}^{\theta}(\sin\varphi\operatorname{cosec}\theta - \sqrt{(\sin^2\varphi-\sin^2\theta\cos^2\varphi)}\,d\varphi\right]d\theta = \frac{4}{\pi^2} + \frac{8}{\pi^2}\int_0^{\frac{1}{2}\pi}\int_{\phi'}^{\theta}\frac{\tan^2\varphi\sec\varphi\,d\theta\,d\varphi}{\sqrt{(\tan^2\varphi-\sin^2\theta)}}.$$

The last term can be expanded and then integrated in series.

143. Proposed by L. C. WALKER, A.M., Professor of Mathematics, Colorado School of Mines, Golden, Col.

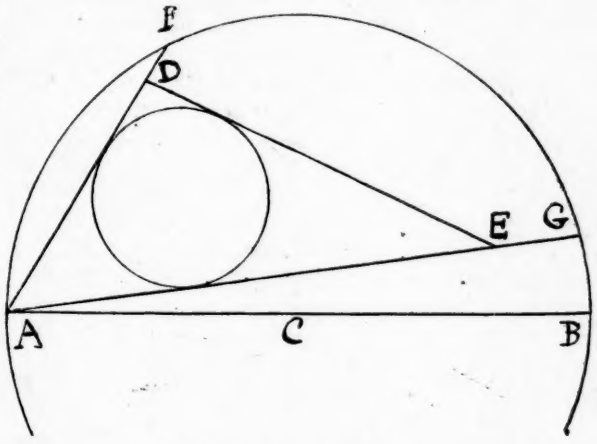
The extremities of two equal lines drrwn from a fixed point in the circumference of a given circle is joined. Find the average area of the circle inscribed in the triangle formed.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, West Va.

Let $C$ be the center of the given circle, $A$ the given point on the circumference, $AC=r$, $AD=AE=x$, $\angle FAB=\theta$, $\angle FAG=2\varphi$. Then $AF=2r\cos\theta=x'$, $DE=2x\sin\varphi$, perimeter $ADE=2x(1+\sin\varphi)$.

Area $ADE=\frac{1}{2}x^2\sin 2\varphi=x^2\sin\varphi\cos\varphi$. Area of inscribed circle$=u$

$$=\frac{\pi(\text{area}ADE)^2}{(\frac{1}{2}\text{perimeter}ADE)^2}=\frac{\pi x^2\sin^2\varphi\cos^2\varphi}{(1+\sin\varphi)^2}.$$




The limits of $\theta$ are 0 and $\frac{1}{2}\pi$; of $\varphi$, 0 and $\theta$; of $x$, 0 and $x'$.

$$\therefore\Delta=\frac{\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_0^{x'}u\,d\theta\,d\varphi\,dx}{\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_0^{x'}d\theta\,d\varphi\,dx}=\frac{\pi}{r(\pi-2)}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_0^{x'}\frac{\sin^2\varphi\cos^2\varphi}{(1+\sin\varphi)^2}d\theta\,d\varphi\,x^2dx$$

$$=\frac{8\pi r^2}{3(\pi-2)}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\frac{\cos^3\theta\sin^2\varphi(1-\sin\varphi)}{(1+\sin\varphi)}d\theta\,d\varphi$$

$$=\frac{4\pi r^2}{3(\pi-2)}\int_0^{\frac{1}{2}\pi}\left(12+\sin\theta\cos\theta-4\cos\theta-5\theta-\frac{8}{1+\tan\frac{1}{2}\theta}\right)\cos^3\theta\,d\theta$$

$$=\frac{4\pi r^2}{3(\pi-2)}\int_0^{\frac{1}{2}\pi}\left(8+\sin\theta\cos\theta-4\cos\theta-5\theta-\frac{4\cos\theta}{1+\sin\theta}\right)\cos^3\theta\,d\theta$$

$$=\frac{\pi r^2(1936-615\pi)}{135(\pi-2)}.$$

144. Proposed F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

In a circular park 400 feet in diameter are four *equal* circular ponds of *variable* diameter. What is the probability that a sightless person walking in a straight line from the center of the park will step into a pond?

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, West Va.

In this solution we shall assume that a random radius (from center $C$ of circular park of radius $r$) does not intersect more than one pond on the same side of the center. The required chance, $p$, then is four times the chance that a random radius will intersect a pond lying wholly upon the surface of a quadrant of the given circle.

Let $P$ be center of pond, $x$ its radius, $\theta$=angle between tangents from $C$ to pond, $\varphi$=angle at $P$ between radii to points of contact. Let $CP=y$. Then $\theta=2\sin^{-1}(x/y)$, $\varphi=\frac{1}{2}\pi-2\sin^{-1}(x/y)$. The limits of $x$ are 0 and $r(\sqrt{2}-1)=x'$; of $y$, $x\sqrt{2}$ and $r-x$.

$$\therefore p=4\frac{\int_0^{x'}\int_{x\sqrt{2}}^{r-x}\theta\varphi\,dxydy}{\int_0^{x'}\int_{x\sqrt{2}}^{r-x}2\pi\varphi\,dxydy}.\quad 2\pi\int_0^{x'}\int_{x\sqrt{2}}^{r-x}\varphi dxydx$$

$$=\frac{\pi}{2}\int^{x'}\left[\pi(r-x)^2-4(r-x)^2\sin^{-1}\left(\frac{x}{r-x}\right)-4x\sqrt{(r^2-2rx)}+4x^2\right]dx$$

$$=\frac{\pi r^3}{18}(3\pi-28+16\sqrt{2}).$$

$$\therefore p=\frac{72}{\pi r^3(3\pi-28+16\sqrt{2})}\int_0^{x'}\int_{x\sqrt{2}}^{r-x}\theta\varphi\,dxydy$$

$$=\frac{36}{\pi r^3(3\pi-28+16\sqrt{2})}\int_0^{x'}\left[\pi(r-x)^2\sin^{-1}\left(\frac{x}{r-x}\right)+\pi x^2+\pi x\sqrt{(r^2-2rx)}\right.$$

$$\left.-4(r-x)^2\left\{\sin^{-1}\left(\frac{x}{r-x}\right)\right\}^2-9x\sqrt{(r^2-2rx)}\sin^{-1}\left(\frac{x}{r-x}\right)-8x^2\log\left(\frac{r-x}{x\sqrt{2}}\right)\right]dx$$

$$=\frac{4}{(3\pi-28+16\sqrt{2})}\left[4\sqrt{2}-3+\frac{8}{\pi}-\frac{8\sqrt{2}}{\pi}+\frac{8}{\pi}\log(2-\sqrt{2})\right].$$

## MISCELLANEOUS.

138. Proposed by L. C. WALKER, A.M., Professor of Mathematics, Colorado School of Mines, Golden, Col.

Find an invariant of the *third degree* in the coefficients of a ternary cubic.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, West Va.

Let $U=ax^4+by^4+cz^4+4(a_2x^3y+a_3x^3z+b'_3y^3z+b_1y^3x+c_1z^3x+z_3z^3y)$

$$+6(dy^2z^2+ez^2x^2+fx^2y^2)+12xyz(lx+my+nz)$$

be the ternary quartic. The required invariant written out in full is

$$\frac{d^4u}{dx^4}\cdot\frac{d^4u}{dy^4}\cdot\frac{d^4u}{dz^4}-4\left(\frac{d^4u}{dx^4}\cdot\frac{d^4u}{dy^3\,dz}\cdot\frac{d^4u}{dz^3\,dy}+\frac{d^4u}{dy^4}\cdot\frac{d^4u}{dz^3\,dx}\cdot\frac{d^4u}{dx^3\,dz}+\frac{d^4u}{dz^4}\cdot\frac{d^4u}{dx^3\,dy}\cdot\frac{d^4u}{dy^3\,dx}\right)$$

$$+3\left[\frac{d^4u}{dx^4}\left(\frac{d^4u}{dy^2\,dz^2}\right)^2+\frac{d^4u}{dy^4}\left(\frac{d^4u}{dx^2\,dz^2}\right)^2+\frac{d^4u}{dz^4}\left(\frac{du^4}{dx^2\,dy^2}\right)\right]$$

$$+4\left(\frac{d^4u}{dx^3dy}\cdot\frac{d^4u}{dy^3\,dz}\cdot\frac{d^4u}{dz^3\,dx}+\frac{d^4u}{dx^3\,dz}\cdot\frac{d^4u}{dy^3dx}\cdot\frac{d^4u}{dz^3\,dy}\right)-12\left(\frac{d^4u}{dx^3dy}\cdot\frac{d^4u}{dxdydz^2}\cdot\frac{d^4u}{dy^2\,dz^2}\right.$$

$$\left.+\frac{d^4u}{dx^3\,dz}\cdot\frac{d^4u}{dx\,dy^2dz}\cdot\frac{d^4u}{dy^2\,dz^2}+\frac{d^4u}{dy^3dx}\cdot\frac{d^4u}{dx\,dy\,dz^2}\cdot\frac{d^4u}{dz^2\,dx^2}+\frac{d^4u}{dy^3\,dz}\cdot\frac{d^4u}{dx^2\,dy\,dz}\cdot\frac{d^4u}{dz^2\,dx^2}\right.$$

$$\left.+\frac{d^4u}{dz^3\,dx}\cdot\frac{d^4u}{dx\,dy^2\,dz}\cdot\frac{d^4u}{dx^2\,dy^2}+\frac{d^4u}{dz^3\,dy}\cdot\frac{d^4u}{dx^2\,dy\,dz}\cdot\frac{d^4u}{dx^2\,dy^2}\right)+12\left(\frac{d^4u}{dx^2dydz}\cdot\frac{d^4u}{dy^3dx}\cdot\frac{d^4u}{dz^3\,dx}\right.$$

$$\left.+\frac{d^4u}{dx\,dy^2dz}\cdot\frac{d^4u}{dz^3\,dy}\cdot\frac{d^4u}{dx^3dy}+\frac{d^4u}{dx\,dy\,dz^2}\cdot\frac{d^4u}{dx^3\,dz}\cdot\frac{d^4u}{dy^3\,dz}\right)+12\left[\frac{d^4u}{dy^2\,dz^2}\cdot\left(\frac{d^4u}{dx^2\,dy\,dz}\right)^2\right.$$

$$\left.+\frac{d^4u}{dz^2\,dx^2}\cdot\left(\frac{d^4u}{dx\,dy^2\,dz}\right)^2+\frac{d^4u}{dx^2\,dy^2}\cdot\left(\frac{d^4u}{dx\,dy\,dz^2}\right)^2\right]+6\frac{d^4u}{dy^2\,dz^2}\cdot\frac{d^4u}{dy^2\,dx^2}\cdot\frac{d^4u}{dx^2\,dz^2}$$

$$-12\,\frac{d^4u}{dx^2\,dy\,dz}\cdot\frac{d^4u}{dx\,dy^2\,dz}\cdot\frac{d^4u}{dx\,dy\,dz^2}.$$

$$\therefore abc-4(ab_3c_2+bc_1a_3+ca_2b_1)+3(ad^2+be^2+cf^2)+4(a_2b_3c_1+a_3b_1c_2)$$

$$-12(a_2nd+a_3md+b_1ne+b_3le+c_1mf+c_2lf)+12(lb_1c_1+mc_2a_2+na_3b_3)+12(dl^2$$

$$+em^2+fn^2)+6def-12lmn.$$

139. Proposed by L. C. WALKER, A.M., Professor of Mathematics, Colorado School of Mines, Golden, Col.

Given the roots of a binary cubic, to find the roots of its two inbependent covariants.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, West Va.

Let $ax^3+3bx^2y+3cxy^2+dy^3=0$ be the cubic. The two independent covariants are,

$$(ac-b^2)x^2+(ad-bc)xy+(bd-c^2)y^2=0. \qquad (1)$$

$$(ad^2-3bcd+2c^3)x^3+3(2b^2d-acd-bc^2)x^2y+3(2ac^2-abd-b^2c)xy^2 + (a^2d-3abc+2b^3)y^3=0. \qquad (2)$$

Let $x/y=z$, then $az^3+3bz^2+3cz+d=0$ is the cubic. Let $\alpha$, $\beta$, $\gamma$ be the gamma roots of this cubic. Then

$$A=(\alpha-\beta)^2+(\beta-\gamma)^2+(\gamma-\alpha)^2=2(\alpha^2+\beta^2+\gamma^2)-2(\alpha\beta+\alpha\gamma+\beta\gamma) =-(18/a^2)(ac-b^2);$$

$$B=2[\alpha(\beta-\gamma)^2+\beta(\gamma-\alpha)^2+\gamma(\alpha-\beta)^2]=-(18/a^2)(ad-bc);$$

$$C=\alpha^2(\beta-\gamma)^2+\beta^2(\gamma-\alpha)^2+\gamma^2(\alpha-\beta)^2=-(18/a^2)(bd-c^2).$$

$\therefore$ The quadratic covariant is $Az^2+Bz+C=0$. The roots of this equation are $[B\pm\sqrt{(B^2-4AC)}]/2A$. Substituting the values of $A$, $B$, $C$ in terms of $\alpha$, $\beta$, $\gamma$, the roots are

$$\frac{\alpha(\beta-\gamma)^2+\beta(\gamma-\alpha)^2+\gamma(\alpha-\beta)^2\pm(\alpha-\beta)(\gamma-\alpha)(\beta-\gamma)\sqrt{(-3)}}{(\alpha-\beta)^2+(\gamma-\alpha)^2+(\beta-\gamma)^2}.$$

$$D=2\beta\gamma-\alpha\beta-\alpha\gamma=3\beta\gamma-(\beta\gamma+\alpha\beta+\alpha\gamma)=3\beta\gamma-3c/a;$$

$$E=2\gamma\alpha-\beta\gamma-\beta\alpha=3\gamma\alpha-3c/a;$$

$$F=2\alpha\beta-\gamma\alpha-\gamma\beta=3\alpha\beta-3c/a;$$

$$G=\alpha+\beta-2\gamma=\alpha+\beta+\gamma-3\gamma=-3b/a-3\gamma;$$

$$H=\beta+\gamma-2\alpha=-3b/a-3\alpha;$$

$$K=\gamma+\alpha-2\beta=-3b/a-3\beta.$$

$$\therefore DEF=27(\alpha\beta-c/a)(\gamma\alpha-c/a)(\beta\gamma-c/a)=(27/a^3)(ad^2-3bcd+2c^3);$$

$$GHK=-27(\alpha+b/a)(\beta+b/a)(\gamma+b/a)=(27/a^3)(a^2d-3abc+2b^3);$$

$$DEG+DFK+EFG=-27[(\beta\gamma-c/a)(\gamma\alpha-c/a)(\gamma+b/a)+(\beta\gamma-c/a) \times(\alpha\beta-c/a)(\beta+b/a)+(\gamma\alpha-c/a)(\alpha\beta-c/a)(\alpha+b/a)] =-(81/a^3)(2b^2d-acd-bc^2);$$

$$DGK+EGH+FHz=27[(\beta\gamma-c/a)(\gamma+b/a)(\beta+b/a)+(\gamma\alpha-c/a)(\alpha+b/a) \times(\gamma+b/a)+(\alpha\beta-c/a)(\alpha+b/a)(\beta+b/a)] =(81/a^3)(2ac^2-abd-b^2c).$$

$\therefore$ The cubic covariant is

$$DEFz^3+(DEG+DFK+EFH)z^2+(DGK+EGH+FHK)z+GHK=0.$$

$$\therefore z^3+(G/F+K/E+H/D)z^2+(GK/EF+GH/DF+HK)DE)z+GHK/DEF=0.$$

$$\therefore (z+G/F)(z+K/E)(z+H/D)=0.$$

$\therefore$ The roots are $-G/F$, $-K/E$, $-H/D$, or

$$\frac{2\gamma-\alpha-\beta}{2\alpha\beta-\gamma\alpha-\gamma\beta}, \quad \frac{2\beta-\alpha-\gamma}{2\gamma\alpha-\beta\gamma-\beta\alpha}, \quad \frac{2\alpha-\beta-\gamma}{2\beta\gamma-\alpha\beta-\alpha\gamma}.$$

## PROBLEMS FOR SOLUTION.

### ALGEBRA.

185. Proposed by L. E. DICKSON, Ph. D., Assistant Professor of Mathematics, The University of Chicago.

Without introducing radicals, eliminate $x$ and $y$ from the equations

$$ax^2+bx+c=0,\ ay^2+by+d=0,\ ax^2y^2+bxy+e=0.$$

186. Proposed by L. E. DICKSON, Ph. D., Assistant Professor of Mathematics, The University of Chicago.

Eliminate $x$ and $y$ from the equations

$$ax^3+bx^2+cx+d=0,$$
$$ay^3+by^2+cy+e=0,$$
$$ax^3y^3+bx^2y^2+cxy+f=0,$$

the eliminant to be rational in $d$, $e$, $f$.

### GEOMETRY.

207. Proposed by W. W. HART, University High School, Chicago, Ill.

According to Gauss the circumference of a circle can be divided into $n$ equal parts by ruler and compass when and only when $n$ is a prime of the form $2^{2^p}+1$.

The following construction gives good partial results for $n$ equals *any* integer. If $AB$ is the diameter of the circle, and $C$ is the vertex of the equilateral triangle $ABC$, and if $D$ is a point on $AB$ at the distance $2AB/n$ from $A$, then draw the line $CD$ cutting the circle at $E$ and $F$; $E$ being the more remote from $C$. $AE=1/n$ circumference approximately. For low values of $n$ this method is very practical; is it practical in general? How great is the error?

208. Proposed by W. J. GREENSTREET, A. M., Editor of The Mathematical Gazette, Stroud, England.

Tangents drawn to two confocal parabolae from a point on the common tangent intersect at the same angle as the axes of the parabolae.

## MECHANICS.

163. Proposed by W. J. GREENSTREET, A. M., Editor of The Mathematical Gazette, Stroud, England.

A particle $A$, mass $m$, rests on a smooth horizontal plane and is attached by two inelastic strings to masses $m_1$, $m_2$ at points $B$ and $C$ such that $BAC$ is a right angle. If a blow is given $A$ at an angle $\theta$ to $AB$, find the initial direction of motion of $m$, and equations for initial motion of the particles $m_1$ and $m_2$.

## NOTES.

The students of The University of Chicago have organized an Elementary Mathematics Club.

Professor J. B. Shaw of Kenyon College, has accepted a call to Millikan University at Decatur, Ill.

Mr. F. E. Knowles has been appointed to an instructorship in mathematics at the University of Oklahoma.

Mr. Floid Field has been appointed Instructor in Mathematics at the Academy of Northwestern University.

Mr. W. O. Beal has been appointed to an instructorship in mathematics at the Illinois College, Jacksonville, Ill.

Mr. J. W. Brister has been elected Associate Professor of Mathematics at the University of Nashville, Nashville, Tenn.

Dr. C. M. Mason (Göttingen) has been appointed Instructor of Mathematics at the Massachusetts Institute of Technology.

Professor H. B. Leonard of the Illinois Wesleyan University will spend this year in doing graduate work at The University of Chicago.

Professor G. B. M. Zerr is superintending the erection of a large electrochemical plant for the manufacture of bleach at Parsons, W. Va.

Mr. A. B. Pierce, formerly Instructor in Mathematics at the University of California, took his Doctor's degree at the University of Zürich, Switzerland, this summer.

### ERRATA IN THIS NUMBER.

Page 219, middle, *for* al *read* all; page 220, line 6, insert (

Page 221, in (10), *read* $(2-b)t^3$; page 222, line 12, *read* It remains...belongs

Page 223, line 11, *read* $c=\pm 1$; line 17, *read* reduce to

Page 224, in (23), *read* $C_r=3S_r-$; *after* $y^3-y^2+y+1$ *put* $=0$; in (24), *for* $S_{k+1}$ *read* $S_{k-1}$.

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## CONCERNING SIMPLE CONTINUED FRACTIONS.

By PROFESSOR THOMAS E. McKINNEY, Marietta, Ohio.

1. In his *Lecons sur la théorie des fonctions*, Chapter II, Borel gives an account of Liouville's demonstration of the existence of non-algebraic numbers. As an application of the theorem on which this demonstration rests he considers the approximation of incommensurable numbers by the method of simple continued fractions and is led to an important proposition in the elementary theory of the subject. As his treatment of this proposition is different from that adopted in any standard text on Algebra—in fact the proposition in the general form is not is not in any Algebra with which I am acquainted—it is proposed to give with some modification of arrangement and detail Borel's account of the theorem and its application to continued fractions.

2. Denote by $\xi$ a real number satisfying an irreducible equation $f(x)=0$ with integral coefficients, of degree $n$ in $x$. Let $p/q$ be a rational fraction in its lowest terms and let both $p/q$ and $\xi$ lie in a certain interval $\alpha$, $\beta$, where $\alpha$ and $\beta$ are both finite but otherwise arbitrary. Then the following proposition may be established:

THEOREM. *A positive number $M$ can be determined such that whatever be the number $p/q$ in the interval $\alpha$, $\beta$ the inequality*

$$(1) \quad \left| \frac{p}{q} - \xi \right| > \frac{1}{Mq^n}$$

*is satisfied.*

Since $f(x)$ is a polynomial in $x$ it is finite and continuous in the interval

$\alpha$, $\beta$ and has a derivative which is also a polynomial and consequently finite in the same interval. Hence there is a positive number $M$ such that

$$(2) \quad |f'(x)| < M$$

in the entire interval $\alpha$, $\beta$.

Inasmuch as $f(x)$ is a polynomial with integral coefficients,

$$f(p/q) = A/q^n,$$

where $A$ is an integer. If $p/q$ is not a root of the equation $f(x)=0$ then $|A| \geqq 1$, and consequently

$$(3) \quad \left| f\left(\frac{p}{q}\right) \right| \geqq \frac{1}{q^n}.$$

Now by Taylor's formula, since $f(\xi)=0$,

$$f(p/q) = (p/q - \xi) f'[\xi + \theta(p/q - \xi)], \quad 0 < \theta < 1.$$

Since $\xi + \theta(p/q - \xi)$ lies in the interval $\alpha$, $\beta$ we may apply (2). Hence

$$|f(p/q)| < |p/q - \xi| M.$$

Then by inequality (3) we have (1) after a slight reduction.

3. Since $\xi$ is a real number it may be expressed in the form of a continued fraction, in the usual notation

$$\xi = (a_0, a_1, \ldots, a_m, \xi_{m+1}),$$

where $\xi_{m+1} \geqq 1$, while $a_1, a_2, \ldots, a_m$ are positive integers, not zero. Denote the $(i+1)$th convergent of this continued fraction by $p_i/q_i$. Then

$$\xi = \frac{p_m \xi_{m+1} + p_{m-1}}{q_m \xi_{m+1} + q_{m-1}}$$

and

$$\frac{p_m}{q_m} - \xi = -\frac{p_{m-1} q_m - p_m q_{m-1}}{q_m^2 \left(\xi_{m+1} + \frac{q_{m-1}}{q_m}\right)} = \frac{(-1)^{m+1}}{q_m^2 \left(\xi_{m+1} + \frac{q_{m-1}}{q_m}\right)}.$$

Since $a_m > 0$, $q_{m-1}/q_m > 0$, and, consequently,

$$(4) \quad \left| \frac{p_m}{q_m} - \xi \right| < \frac{1}{\xi_{m+1} q_m^2}.$$

Replacing $p$ and $q$ in inequality (1) by $p_m$ and $q_m$ respectively, and comparing the results with inequality (4) we have